7 Equations of Fluid Motion

7.1 INTRODUCTION

In the previous chapters, we have examined characteristics of static fluids and developed a coarse model of a mechanical energy balance on a fluid moving in a conduit. Beginning in this chapter, we will look at fluid behavior in more detail. In developing models for fluid flow, we use mass conservation and Newton's second law of motion to derive relationships between the velocity field and the physical mechanisms driving and restraining flow. These mass and force balances will be used in later chapters to describe many different flows and the heat and mass transport in them.

These models for mass and momentum conservation are extremely useful in understanding the behavior of flow, especially in that each term in the equations represents some physical phenomenon and each equation shows the interaction of those phenomena. As we have seen with the diffusion of heat and mass, understanding these interactions is important for gaining insight into the behavior of the transport phenomena, even when we do not explicitly solve the equation. Thus, a brief introduction to the derivation of these governing equations and a recognition of their physical meaning is vital to understanding why fluids behave as they do. Studying fluid mechanics without this beginning is like studying basic Newtonian physics without calculus; we could take that approach, but it is an unnecessary obstacle to learning.

The derivation of these governing equations is facilitated by looking at these flows from a particular point of view. We use the approach, termed the *Eulerian* point of view, of observing flow through and forces acting on a small (generally fixed in space) control volume to derive the governing equations. Eulerian modeling includes not only the forces and flows interacting with a small control volume, but also the local accumulation of mass or momentum at a fixed point in space.[1] This method is different from the *Lagrangian* point of view, which follows the behavior of a small packet of fluid as it moves through space under the influence of various effects.

7.2 CONSERVATION OF MASS

To model the conservation of mass in a flowing fluid, we begin by examining mass moving through an infinitesimal Cartesian control volume, fixed in space, with a differential volume, $d\forall = dx\,dy\,dz$. The mass balance requires that the rate of change of mass within the control volume be the difference between the rates at which mass enters and exits the volume:

$$\begin{aligned}\left(\text{mass accumulation or depletion rate}\right) = \left(\text{rate of mass entering}\right) \\ -\left(\text{rate of mass leaving}\right).\end{aligned} \tag{7.1}$$

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Consider the two-dimensional control volume in Cartesian coordinates shown in Figure 7.1. (We will show the details of the derivations of mass conservation and (in the next section) the momentum balance in a two-dimensional velocity field, but will also present the three-dimensional versions at the end of the process.) The rate at which mass flows across the volume interfaces is a product of the density, the magnitude of the velocity normal to the interface, and the area of that interface:

$$\dot{m} = \rho \left| \bar{V} \right| A, \tag{7.2}$$

where $\dot{m}$ has the units kg/s. Referring to Figure 7.1, we can rewrite Eq. (7.1) as

$$\frac{\partial m}{\partial t} = \left(\dot{m}_x + \dot{m}_y \right) - \left(\dot{m}_{x+dx} + \dot{m}_{y+dy} \right), \tag{7.3}$$

where the three terms are the mass storage in the volume, the mass flow rate in, and the mass flow rate out. Alternately, we can write this mass balance in terms of the change in mass flow rate across the volume in each direction:

$$\underset{\substack{\text{mass}\\\text{stored}}}{\frac{\partial m}{\partial t}} = \underset{\substack{\text{mass rate}\\\text{of change}\\\text{in x}}}{\left(\dot{m}_x - \dot{m}_{x+dx} \right)} + \underset{\substack{\text{mass rate}\\\text{of change}\\\text{in y}}}{\left(\dot{m}_y - \dot{m}_{y+dy} \right)}. \tag{7.4}$$

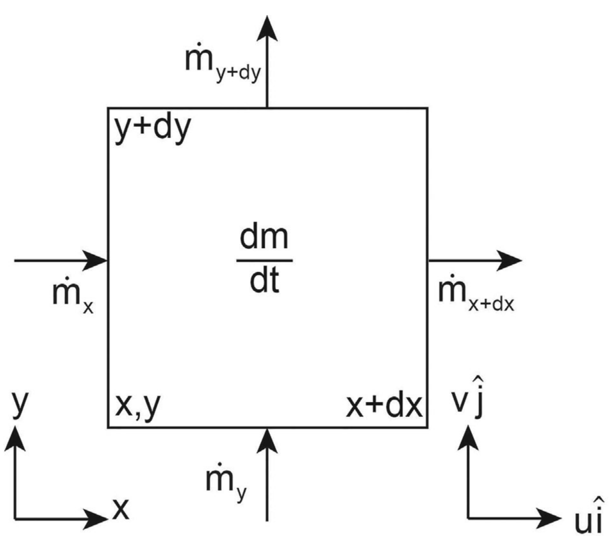

FIGURE 7.1 Mass flow rates and storage in a two-dimensional Cartesian control volume.

The fluid velocity is a vector field, which comprises three components:

$$\bar{V} = u\hat{i} + v\hat{j} + w\hat{k} \quad (7.5)$$

in Cartesian coordinates and, in cylindrical coordinates,

$$\bar{V} = u\hat{r} + v\hat{\theta} + w\hat{z}. \quad (7.6)$$

Using the Cartesian definition of the mass flow rates across the surfaces at x and y and of the velocity vector in Eqs. (7.2) and (7.5), we get

$$\dot{m}_x = \rho u_x A_x = \rho u_x dydz \quad \text{and} \quad \dot{m}_y = \rho v_y A_y = \rho v_y dxdz. \quad (7.7)$$

(We have assumed the control volume has a depth dz into the page.) A Taylor series approximation (truncated to two terms) can be used to obtain estimates of the mass flow rates in both directions in close vicinity of positions x and y:

$$\dot{m}_{x+dx} = \rho u_{x+dx} A_{x+dx} \quad \text{and} \quad \dot{m}_{y+dy} = \rho v_{y+dy} A_{x+dy}$$

$$\dot{m}_{x+dx} = \left[(\rho u)_x + \left.\frac{\partial \rho u}{\partial x}\right|_x dx \right] dydz \quad \text{and} \quad \dot{m}_{y+dy} = \left[(\rho v)_y + \left.\frac{\partial \rho v}{\partial y}\right|_y dy \right] dxdz. \quad (7.8)$$

The mass storage term can be written as

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho d\forall) = \frac{\partial \rho}{\partial t}(dxdydz). \quad (7.9)$$

Combining relationships (7.7)–(7.9) in the mass flow balance (7.4) produces

$$\frac{\partial \rho}{\partial t} dxdydz = -\left[\left.\frac{\partial (\rho u)}{\partial x}\right|_x dx \right] dydz - \left[\left.\frac{\partial (\rho v)}{\partial y}\right|_y dy \right] dxdz.$$

Dividing by the differential volume gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0, \quad (7.10)$$

or in three dimensions

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0. \quad (7.11)$$

The first term in Eqs. (7.10) and (7.11) is the local storage of mass in a constant volume. The other terms are the change in the mass fluxes (ρu, ρv, ρw) across the control volume. In cylindrical coordinates, where the velocity field is represented by Eq. (7.6), a similar process gets

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial(r\rho u)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v)}{\partial \theta}+\frac{\partial(\rho w)}{\partial z}=0. \tag{7.12}$$

If the density is constant and uniform, then Eq. (7.11) is reduced to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{7.13}$$

in Cartesian coordinates and, in cylindrical coordinates,

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0. \tag{7.14}$$

This approximation holds true in incompressible fluids (such as liquids), where density is at most a weak function of pressure, and in gases with small pressure differences. These mass balance equations, Eqs. (7.11) and (7.12), are known as the *mass conservation equation* or the *continuity equation*. The first name is obvious from our derivation; the second is because this equation represents a constraint on the momentum equations (in the next section), which keeps the fluid continuous with no gaps or overlaps.

To illustrate the relationship between the two mass flux terms in Eq. (7.10), Figure 7.2 shows a volume with velocity in from the left side (u_1) and out the right (u_2), with some flow through the top surface at velocity v_2 and none through the bottom ($v_1 = 0$). If $u_1 > u_2$, then the flow decelerates in x and $\partial u/\partial x < 0$. From Eq. (7.13) with $\partial w/\partial z = 0$, we see that $\partial v/\partial y > 0$ or $v_2 > v_1$. In this case, the flow enters from the left and splits into two different directions; mass leaving out the top of the volume ($v_2 > 0$) slows the horizontal velocity. In the opposite case, if $u_1 < u_2$ (accelerating

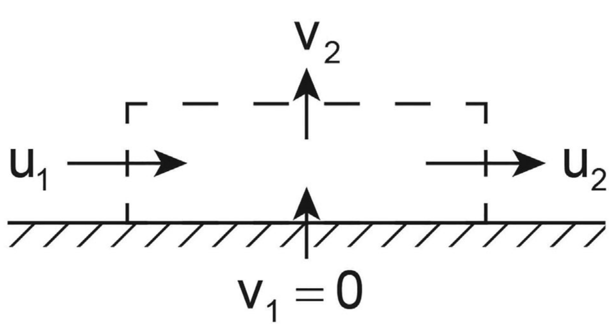

FIGURE 7.2 Control volume illustrating conservation of mass in two dimensions.

flow), then $\partial u/\partial x > 0$ and $\partial v/\partial y < 0$, so $v_2 < v_1$. To accomplish this increased flow rate in the x direction, mass must be entrained into the volume through the top with a downward velocity, $v_2 < 0$.

Beginning with Eq. (7.13) for two-dimensional steady flow ($w = 0$), we can define the *streamfunction*, $\psi(x, y)$, in Cartesian coordinates as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{7.15}$$

This quantity is defined thus to satisfy the continuity equation identically:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0.$$

The total differential of $\psi(x,y)$ is

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\,dx + u\,dy. \tag{7.16}$$

We also define a *streamline* as a line in a velocity field everywhere tangent to the flow, as shown in the example flow field in Figure 7.3 (a). This tangency condition means that velocity has no *normal* component on a streamline and so streamlines are impermeable. The slope of the streamline can be defined as

$$\frac{dy}{dx} = \frac{v}{u}, \tag{7.17}$$

where the rise over run is determined by the ratio of velocity components. Eqs. (7.16) and (7.17) combined gives

$$-v\,dx + u\,dy = d\psi = 0,$$

which shows that the value of the streamfunction is uniform along a streamline.

The fact that no fluid crosses a streamline (a line of constant ψ) is useful for displaying the distribution of the volume flow rate, $Q = \left|\bar{V}\right| A$, in a flow field. Any two streamlines form a *stream tube*, and we find that the flow rate is the same across any line drawn from one streamline to the other. This result is illustrated in Figure 7.3 (b) by calculating Q across the two lines, A and B. For line A, the only normal velocity component is u, so

$$Q_A = \int_{y_1}^{y_2} u\,dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} dy = \psi_2 - \psi_1,$$

For line B, with no horizontal component of velocity,

$$Q_B = \int_{x_2}^{x_1} v\,dx = -\int_{x_1}^{x_2} v\,dx = \int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} dx = \psi_2 - \psi_1.$$

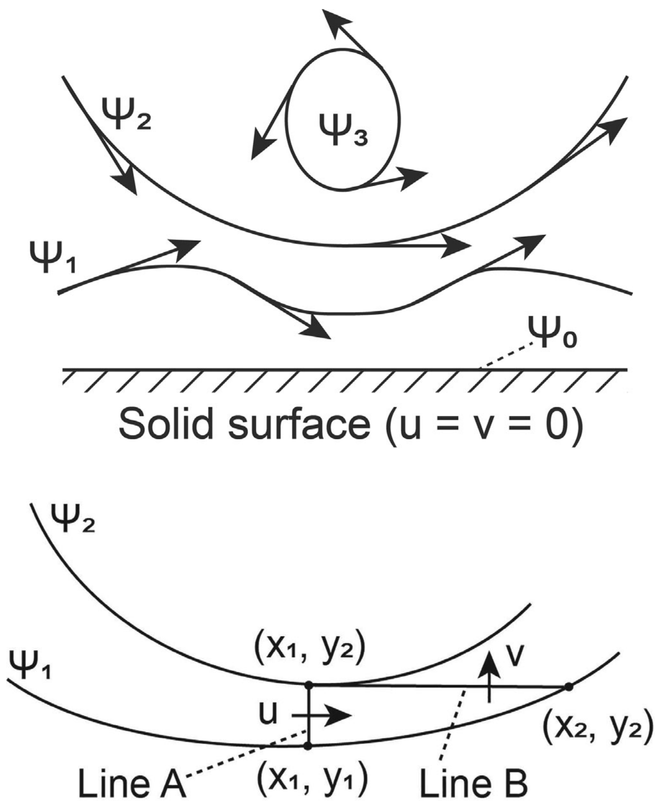

FIGURE 7.3 (a) Streamlines (lines of constant ψ) in a flow field, showing tangent velocity vectors. The surface also acts as a streamline, as no flow passes through it. (b) Flow through a stream tube between two streamlines (ψ_1 and ψ_2).

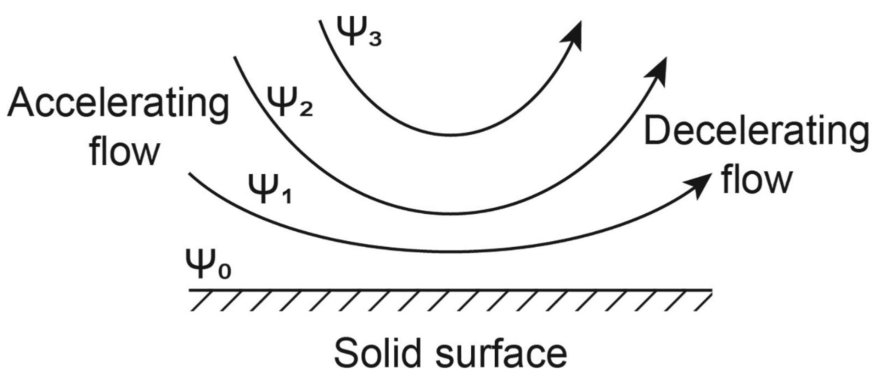

FIGURE 7.4 Streamlines in a flow field moving from left to right. The flow first accelerates as streamlines converge, then decelerates as they diverge.

We see that $Q_A = Q_B$, so the volume flow rate does not change along the stream tube constrained by ψ_1 and ψ_2. The value of the volume flow rate, Q, is also the difference of the streamfunction values for the two streamlines.

Figure 7.4 shows a family of streamlines representing a flow field moving left to right. Moving from the left, the streamlines converge then diverge. As they move closer together, the cross sectional areas of the stream tubes decrease, causing an increase in the velocities along the tubes. When they then diverge farther downstream, those areas increase and the flow decelerates.

7.3 MOMENTUM BALANCE: THE NAVIER–STOKES EQUATIONS

In this section, we apply Newton's second law of motion to a fluid control volume to derive a momentum balance equation. This balance and the continuity equation (7.13) form a basis for a description of the mechanisms controlling fluid motion.

In most early physics courses, Newton's second law is written as

$$\sum \bar{F} = m\bar{a},$$

where the mass is constant and $\bar{a}$ is the acceleration vector. A more general version is that the sum of the forces causes a change in the momentum vector, $\bar{\Pi}$, with time:

$$\sum \bar{F} = \frac{d\bar{\Pi}}{dt}. \tag{7.18}$$

We can write the momentum field as

$$\bar{\Pi}(x, y, t) = m\bar{V},$$

and its total derivative in two-dimensional Cartesian coordinates as

$$d\bar{\Pi} = \frac{\partial(m\bar{V})}{\partial t} dt + \frac{\partial(m\bar{V})}{\partial x} dx + \frac{\partial(m\bar{V})}{\partial y} dy. \tag{7.19}$$

The total change in momentum with time is then

$$\begin{aligned} \frac{d\bar{\Pi}}{dt} &= \frac{\partial(m\bar{V})}{\partial t}\frac{dt}{dt} + \frac{\partial(m\bar{V})}{\partial x}\frac{dx}{dt} + \frac{\partial(m\bar{V})}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial(m\bar{V})}{\partial t} + u\frac{\partial(m\bar{V})}{\partial x} + \frac{\partial(m\bar{V})}{\partial y}, \end{aligned} \tag{7.20}$$

where $u = dx/dt$ and $v = dy/dt$. This form of the total time derivative in a flow field is known as the substantial, or *material*, *derivative*, which is usually written for any transported quantity as