$$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\text{storage}} + \underbrace{u\frac{\partial(\rho\phi)}{\partial x} + v\frac{\partial(\rho\phi)}{\partial y}}_{\text{advection}} = \underbrace{\Gamma\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)}_{\text{diffusion}} + \underbrace{\dot{S}}_{\text{source}}. \quad (10.11)$$

Because these two equations are of the same form, we expect some similarities in behavior, subject to the boundary conditions. In the absence of generation terms, the similarities are governed by the ratio of the thermal diffusivity, α, and the momentum diffusivity, v. That ratio is

$$\frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{v}{\alpha} = \left(\frac{\mu}{\rho}\right)\bigg/\left(\frac{k}{\rho c}\right) = \frac{\mu c}{k} = \text{Pr}, \quad (10.12)$$

where Pr is the Prandtl number.[1] This dimensionless parameter shows us the relative speed of transfer of heat and momentum by diffusion in a given fluid.

Unlike many other dimensionless parameters we discuss in this text, Pr is a material property of the fluid, so we can examine values for a variety of interesting substances (Table 10.1). First, we notice that the order of magnitude of the specific heat of most materials is in a range between $O(10^2)$ and $O(10^3)$ J/kgK, so the order of magnitude of Pr is primarily governed by the relative values of the more widely ranging quantities, viscosity, μ, and thermal conductivity, k. Gases tend to have Pr just less than one, as their thermal diffusivities are slightly higher than their kinematic viscosities. Water is somewhat more viscous relative to its thermal diffusivity, and $5 \le \text{Pr} \le 8$ between its freezing and boiling points. Liquid polymers are quite different due to their complex molecular structures; they have low thermal conductivities and high effective viscosities and so behave as a fluid with $\text{Pr} \gg 1$. (Magmas, while not engineering fluids, also have large Pr; being molten ceramic mixtures, they have low k and high μ.) On the other hand, liquid metals have a kinematic viscosity on the order of magnitude as water, but possess a much higher thermal diffusivity, so their $\text{Pr} \ll 1$.

10.3 ADVECTION IN RIGID, MOVING MEDIA

The simplest case of advection heat transfer is a moving isothermal body (e.g., the steel slab mentioned in Section 10.2), but perhaps the next simplest is a process in which a rigid body with internal conduction moves out of a hot region and sheds heat to a colder environment as it does so. Examples of this configuration include polymer strands leaving an extruder and metal rods undergoing continuous induction hardening. In many cases, we can approximate the heat transfer as occurring in one

TABLE 10.1
Estimates of Prandtl Numbers for Several Materials Classes

Pr	≈ 0.01	0.5–1	5–8	> 100
materials	liquid metals	gases	water	polymers, magma

dimension (the direction of motion, or the axial direction) and treating heat losses in perpendicular directions as heat sinks. In order for this approximation to be valid, the heat flow in the body must be oriented so that it is mainly in the axial direction. If the heat flow in the direction of motion is much greater than the directions normal to motion, then this one-dimensional approximation is reasonable.

If the moving body can be modeled as one-dimensional, then we can define a control volume over which we can perform an energy balance in order to derive a conservation equation for thermal energy in terms of temperature. Figure 10.5 shows such a control volume, defined by length dx, cross-sectional area A_c, and perimeter p, and in which thermal energy is transferred by conduction (q_x) and advection in the x direction. The amount of energy that is brought into the control volume at location x by bulk solid motion is $\dot{m}\,e_x = \left(\rho V A_c\, cT\right)_x$, where e_x is the specific enthalpy at x and V is the speed of the moving body. The rate at which energy is advected out of the volume at ($x + dx$) can be different and is written as $\dot{m}\,e_{x+dx} = \left(\rho V A_c\, cT\right)_{x+dx}$. Also, heat can be generated in the volume ($\dot{q}$) and also lost to the ambient by convection from the surface. (This convection loss moves normal to the axial motion. To use a simple one-dimensional model, we treat convection as a heat sink, a "negative generation," rather than as a boundary condition in y or z for a two-dimensional or three-dimensional model.) For the rest of this derivation, we will assume mass flow rate, $\dot{m}$, material properties, heat transfer coefficient (h), and geometry do not change along the direction of motion (x).

The first law of thermodynamics (energy is conserved) for this control volume can be written as the usual energy rate balance:

$$\dot{U}_{in} - \dot{U}_{out} + \dot{U}_{gen} = \dot{U}_{stor}, \tag{2.19}$$

where heat flows in the x direction by conduction and advection, and the process is steady state.

$$\begin{aligned} \dot{U}_{in} &= q_x + \dot{m}\,e_x = q_x + \left(\rho V A_c\, cT\right)_x \quad \dot{U}_{gen} = \dot{q}\,A_c\,dx - h\left(p\,dx\right)(T - T_\infty) \\ \dot{U}_{out} &= q_{x+dx} + \dot{m}\,e_{x+dx} = q_{x+dx} + \left(\rho V A_c\, cT\right)_{x+dx} \quad \dot{U}_{stor} = 0. \end{aligned} \tag{10.13}$$

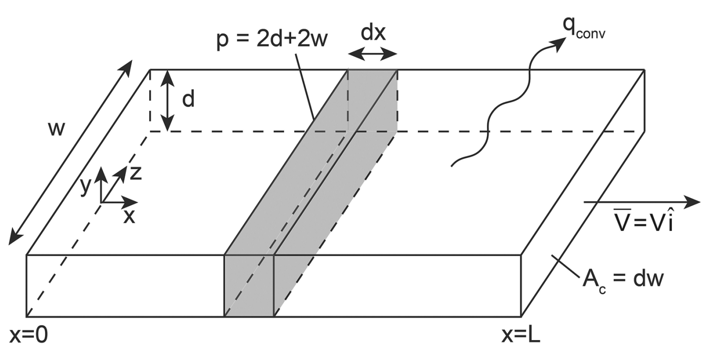

FIGURE 10.5 Control volume (in gray) for analysis of moving rigid body.

Note that only the part of the body that actually exchanges heat with the environment is counted in the perimeter, p. Writing a Taylor series expansion of the specific enthalpy at $(x + dx)$, we obtain

$$e_{x+dx} = e_x + \frac{de_x}{dx}dx + \text{higher order terms.}$$

Using this expression to find the difference between the energy advected in and out of the control volume, we get

$$\dot{m}(e_{x+dx} - e_x) \approx \rho V A_c \, c \frac{dT}{dx}dx. \tag{10.14}$$

Similarly, we can find the change in the diffusion heat transfer over dx:

$$q_{x+dx} - q_x = -\frac{d}{dx}\left(k A_c \frac{dT}{dx}\right)dx. \tag{10.15}$$

Putting these relations into Eq. (2.18), we get the energy conservation equation that describes the temperature along the length of the moving body.

$$\underset{\text{advection}}{\frac{d\left(\rho V A_c c T\right)}{dx}} = \underset{\text{conduction}}{\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right)} - \underset{\substack{\text{convection}\\\text{loss}}}{h\,p\left(T - T_\infty\right)} + \underset{\substack{\text{heat}\\\text{generation}}}{\dot{q}\,A_c} \tag{10.16}$$

It is useful to look carefully at this energy equation to remind ourselves of the physical phenomena that govern it. Eq. (10.16) is equivalent to Eq. (10.10) for a steady, one-dimensional flow. The first term is the change in the thermal energy of a mass as it moves through space, i.e., it is the rate of change of thermal energy advected as it passes a specified point in space. The second term represents the diffusion of thermal energy along the length of the body due to the axial temperature gradient. This conduction term happens independent of the body motion and its magnitude. The third term is the convection heat loss from the outer surface to the environment and the final term is heat generated inside the body. For uniform properties and cross-sectional area, the energy balance in Eq. (10.16) reduces to

$$\underset{\text{advection}}{\rho c V \frac{dT}{dx}} = \underset{\text{conduction}}{k\frac{d}{dx}\left(\frac{dT}{dx}\right)} - \underset{\substack{\text{convection}\\\text{loss}}}{\frac{hp}{A_c}\left(T - T_\infty\right)} + \underset{\substack{\text{heat}\\\text{generation}}}{\dot{q}}. \tag{10.17}$$

The boundary conditions in x for this equation are found by considering the physical configuration. As the body leaves a constant temperature source (e.g., a furnace or extruder), it likely has the temperature of that device. As the body loses heat to the environment, eventually it will cool to the ambient temperature. These boundary conditions are written as

$$T(x=0)=T_o \qquad \text{and} \qquad T(x\to\infty)=T_\infty.$$

At this point, we have an equation that can be solved for temperature as a function of 11 parameters: $T(x, k, A_c, \rho, c, V, h, p, T_\infty, T_o, \dot{q})$. It would simplify matters to reduce the parameter space and produce a more generalized result by nondimensionalizing the equation and boundary conditions, using carefully chosen reference values. We start here by dropping the heat generation term. (This step is not necessary, but most applications do not have this effect and it does simplify the following procedure.) We pick references for the temperature difference, $T(x) - T_\infty$, and for the axial coordinate, x, thus

$$\theta = \frac{T - T_\infty}{T_o - T_\infty} \quad \text{and} \quad \eta = x/l. \tag{10.18}$$

The characteristic length, ℓ, is initially unknown, and its form will be chosen to reduce the number of parameters. Inserting Eq. (10.18) into Eq. (10.17) and rearranging, we get

$$\left[\frac{kA_c\Delta T_o}{l^2}\right]\frac{d^2\theta}{d\eta^2} - \left[\frac{\rho VA_c\, c\Delta T_o}{l}\right]\frac{d\theta}{d\eta} - \left[hp\Delta T_o\right]\theta = 0, \tag{10.19}$$

a second-order differential equation for $\theta = f(\eta)$. We nondimensionalize the equation by dividing by the coefficient of the first term, and setting $\ell = (kA_c/hp)^{1/2}$ and $\mathrm{Pe} = V\ell/\alpha$:

$$\frac{d^2\theta}{d\eta^2} - \left[\frac{V\ell}{\alpha}\right]\frac{d\theta}{d\eta} - \theta = 0, \tag{10.20}$$

we have reduced the problem to one of finding temperature, θ, as a function of the axial coordinate,η, and the Peclet number, Pe. Introducing the Peclet number as the ratio of the speed of advection, V, and an effective thermal diffusion speed, α/ℓ, the energy equation and boundary conditions are then written as

$$\underset{\substack{\text{axial}\\\text{conduction}}}{\frac{d^2\theta}{d\eta^2}} - \underset{\substack{\text{axial}\\\text{advection}}}{Pe\,\frac{d\theta}{d\eta}} - \underset{\substack{\text{convection}\\\text{loss}}}{\theta} = 0 \qquad \theta(\eta=0)=1 \qquad \theta(\eta\to\infty)=0. \tag{10.21}$$

The general solution for Eq. (10.21) is $\theta = \exp(-B\eta)$ where

$$B = \sqrt{(Pe/2)^2 + 1} - (Pe/2). \tag{10.22}$$

We can also find approximate solutions for different physical situations, depending on which terms (i.e., which physical phenomena) dominate the energy conservation equation. There are three interesting cases that are simplifications of the general case [3].

1 **Pe → 0**: In a slow-moving solid (low V), or one with a large thermal diffusivity (α), the advection term is negligible, and relatively little heat is carried

by the motion of the body. The heat flow is then a balance of conduction along the body and the convection heat loss:

$$\frac{d^2\theta_1}{d\eta^2} - \theta_1 = 0 \qquad \theta_1(\eta = 0) = 1 \qquad \theta_1(\eta \to \infty) = 0, \tag{10.23}$$

and the solution is $\theta_1 = e^{-\eta}$ (i.e., $B = 1$). This result is the classic solution for a stationary fin of infinite length [4].

2 **Pe → ∞**: In this situation, the advection term completely overwhelms the other effects and almost all of the heat moving downstream in x is carried by bulk motion. While conduction always occurs, its effect is negligible. If the body is moving fast enough (Pe → ∞), there is not time even for convective losses to occur and the conservation equation (10.21) is:

$$\frac{d\theta_2}{d\eta} \approx 0. \tag{10.24}$$

Integrating this equation and applying the condition at $\eta = 0$, we find $\theta_2 = 1$ ($B = 0$), implying that there is simply no time to lose any thermal energy to the environment, there is effectively no drop in temperature. This result is interesting as a limit, but is not applicable to systems that do have a measurable heat loss.

3 **Pe large, but not infinite**: In this case, advection is much more important than axial conduction (as shown earlier), but is still balanced by heat loss to the ambient. Most practical materials processing problems fall into this case.

$$Pe\,\frac{d\theta_3}{d\eta} + \theta_3 = 0 \qquad \theta_3(\eta = 0) = 1 \tag{10.25}$$

The solution for this regime is $\theta_3 = e^{-\eta/Pe}$ ($B = 1/\text{Pe}$), which is simpler than the general case. The validity of these three approximate cases can be seen in comparison to the exact solution in Figure 10.6.

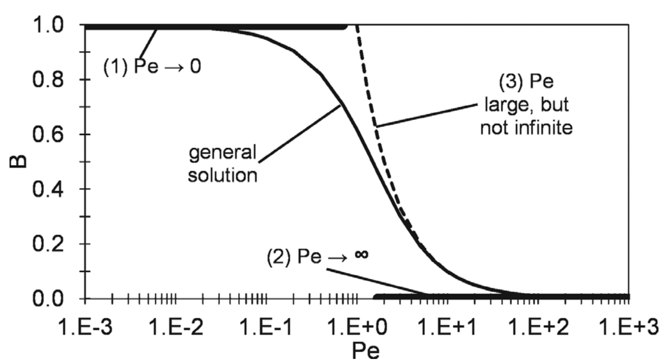

FIGURE 10.6 Coefficient (B) in exponent in solutions of general and special cases of a one-dimensional advection-diffusion problem.